Rapid Calculation of Heat Transfer in Nonsimilar Laminar Incompressible Boundary Layers

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The method of strained coordinates has been extended to treat nonsimilar laminar thermal boundary-layer equations subjected to arbitrarily prescribed freestream velocity. Universal function solutions have been obtained for a very large range of Prandtl numbers. One example is given. Comparison with numerical solutions shows that the present technique is highly accurate, and lends strong support to its use for rapid estimations in most engineering computations.

Nomenclature

C_f	= local friction coefficient, $2\tau_w/\rho U^2$
f	= dimensionless stream function
g	= β-derivative of f
ℓ	= reference length
R	= radius of revolution of an axisymmetric body
r	= radius of cylinder or sphere
Re_x	= Reynold's number, $\rho U x/\mu$
T	= temperature
T_∞	= ambient temperature
T_w	= wall temperature
U_e	= local freestream velocity
U_∞	= reference velocity, also uniform freestream velocity
u	= streamwise velocity component = $(\ell/R)^j\partial\psi/\partial y$
v	= transverse velocity component = $-(\ell/R)^j\partial\psi/\partial x$
x	= streamwise coordinate
y	= transverse coordinate normal to the surface
$\tilde{\beta}$	= Gortler's Principle function, $=2\xi\ell(\ell/R)^{2j}U_\infty/U_e{}^2\mathrm{d}U_e/\mathrm{d}x$
β	= strained value of $\tilde{\beta}$
Pr	= Prandtl Number
Δ_0	= as defined in Eq. (7)
$\epsilon(\tilde{\beta})$	= $2\xi\mathrm{d}\tilde{\beta}/\mathrm{d}\xi$
ϵ'	= $\mathrm{d}\epsilon/\mathrm{d}\tilde{\beta}$
ξ	= transformed streamwise coordinate = $\int_0^x U_e/U_\infty (R/\ell)^{2j}\mathrm{d}x/\ell$
η	= transformed transverse coordinate = $U_e(R/\ell)^j y/(2\nu U_\infty\xi\ell)^{1/2}$
ν	= kinematic viscosity
$\Gamma(x,y)$	= incomplete gamma function
θ	= $(T-T_\infty)/(T_w-T_\infty)$
ρ	= density
τ_w	= wall shear stress
ψ	= stream function = $(2\nu U_\infty\xi\ell)^{1/2}f(\xi,\eta)$
j	= one (1) for axisymmetric flow
j	= zero (0) for two-dimensional flow

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Introduction

THIS paper deals with a simple, yet accurate, method for computing heat transfer coefficients in the laminar boundary layers which do not admit similarity solutions. Although the boundary-layer equations are of parabolic-type, solutions depend strongly on local conditions, and in terms of Görtler-Meksyn variables little memory of upstream history is retained. With these variables, local boundary-layer behavior is characterized by Görtler's Principal function and its streamwise derivatives. Then the local similarity solution provides a good first approximation of the flow field if the local derivative of $\tilde{\beta}$, $\epsilon(\tilde{\beta})$, is small. As will be seen, an even better approximation is provided for the thermal boundary layer.

The asymptotic method developed recently by Kao and Elrod[1] for treating nonsimilar incompressible boundary layers with arbitrary freestream velocity is based on the idea that it is possible to find a shifted value of the local Görtler Principal function $\tilde{\beta}$ such that the corresponding local similarity solution gives a good approximation to the wall shear stress. The present paper extends the analysis as given in Ref. 1, to include laminar thermal boundary layer, as well. Working charts for a very large range of Prandtl number are presented, an illustrative numerical example is given, and the technique is compared with other methods.

Analysis

Upon application of the Görtler-Meksyn transformation to Prandtl's thermal boundary-layer equations for an incompressible laminar two-dimensional or axisymmetric steady flow about a body of revolution,[1] we get

$$f''{}' + ff'' + \tilde{\beta}(1-f'^2) = \epsilon(\tilde{\beta})(f'\partial f'/\partial\tilde{\beta} - f''\partial f/\partial\tilde{\beta}) \quad (1)$$

$$\theta''/Pr + f\theta' = \epsilon(\tilde{\beta})(f'\partial\theta/\partial\tilde{\beta} - \theta'\partial f/\partial\tilde{\beta}) \quad (2)$$

with boundary conditions

$$f(\xi,0) = f'(\xi,0) = 0 \qquad f'(\xi,\infty) = 1 \quad (3)$$

$$\theta(\xi,0) = 1 \qquad \theta(\xi,\infty) = 0 \quad (4)$$

where

$$\epsilon(\tilde{\beta}) = 2\xi\mathrm{d}\tilde{\beta}/\mathrm{d}\xi \quad (5)$$

The nondimensional heat transfer coefficient at the wall is given by

$$Nu/\sqrt{Re} = -(U_e/U_\infty)\ (1/2\xi)^{1/2}\theta'(\beta,0) \tag{6}$$

where $Nu = \bar{h}\ell/k$ and $Re = U_\infty \ell/\nu$. The ℓ is some reference length.

The solutions for Eqs. (1) and (3) are given in Ref. 1. Here we shall concern ourselves mainly with Eq. (2) and boundary conditions (4). θ, f, and $\tilde{\beta}$ are expanded in a power series of ϵ

$$\tilde{\beta} = \beta + \epsilon(\tilde{\beta})\Delta_0(\beta) + \epsilon^2(\tilde{\beta})\Delta_1 + \ldots \tag{7}$$

$$f = f_0(\beta,\eta) + \epsilon(\tilde{\beta})f_1(\beta,\eta) + \epsilon^2(\tilde{\beta})f_2 + \ldots \tag{8}$$

$$\theta = \theta_0(\beta,\eta) + \epsilon(\tilde{\beta})\theta_1(\beta,\eta) + \epsilon^2(\tilde{\beta})\theta_2 + \ldots \tag{9}$$

Upon substituting the preceding expressions into Eqs. (1-4) and equating terms of like power in ϵ, we find that θ_0 and θ_1 satisfies the following two sets of equations

Order unity:

$$\theta_0''/Pr + f_0\theta_0' = 0 \tag{10}$$

$$\theta_0(\beta,0) = 1 \qquad \theta_0(\beta,\infty) = 0 \tag{11}$$

Order ϵ:

$$\theta_1''/Pr + f_0\theta_1' - \epsilon' f_0'\theta_1 = (1-\epsilon'\Delta_0)(f_0'\partial\theta_0/\partial\beta - \theta_0'\partial f_0/\partial\beta) - (1+\epsilon')f_1\theta_0' \tag{12}$$

$$\theta_1(\beta,0) = 0 \qquad \theta_1(\beta,\infty) = 0 \tag{13}$$

where

$$\epsilon' = \mathrm{d}\epsilon/\mathrm{d}\tilde{\beta}$$

The Δ_0 has been found (in Ref. 1) by requiring that $f_1''(\beta, \epsilon', 0) = 0$, so that the local similarity solution f_0, corresponding to the shifted $\beta = \tilde{\beta} - \epsilon\Delta_0$ provides a good approximation (accurate through order ϵ) to the local wall shear stress. For convenience, we have reproduced the results for Δ_0 in Fig. 1 for various combinations of ϵ' and β. Solutions for Eqs. (10, 11) are the usual similarity solutions with $\tilde{\beta}$ replaced by β. These similarity solutions are well-known and can be found, for example, in Ref. 2. A shooting technique as described by Nachsheim and Swigert[3] is used to find $\theta_1'(\beta, \epsilon', \eta = 0)$. Numerical solutions for $Pr = 0.7$, 1.0, 5.0, and 10.0 are obtained and asymptotic formula are developed in the Appendix. In Figs. 2-6, the solutions for $Pr \leq 10$ are presented. For $Pr > 10$, the solutions plotted are obtained either from Eq. (A16), or through interpolation between numerical and asymptotic solutions.

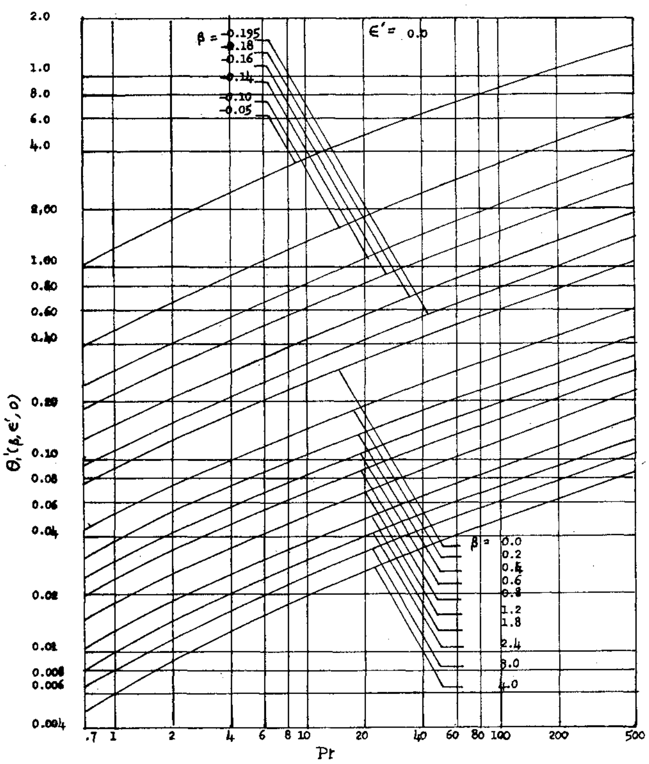

Fig. 2 $\theta_1'(\beta,\epsilon',0)$ vs Pr with $\epsilon' = 0$.

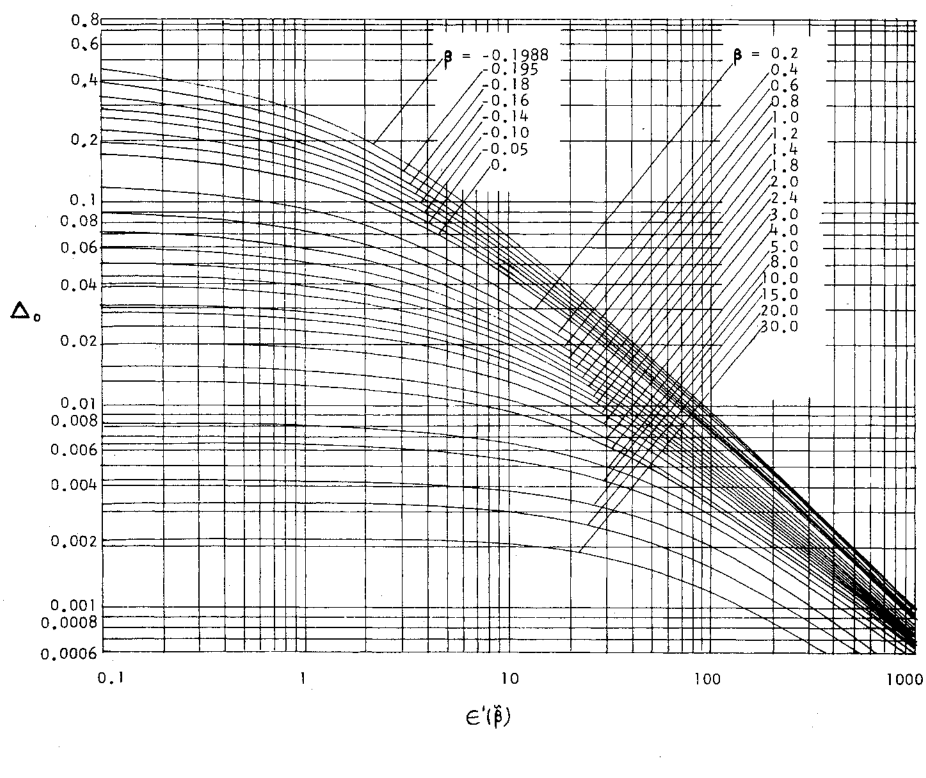

Fig. 1 Δ_0 vs $\epsilon'(\tilde{\beta})$.

Table 1 $T_0'(\beta,\epsilon',0)$

ϵ'/β	0	1	2	3	4	5	6	7	8	9	10	12	14	16
−0.195	−3.55795	−2.13477	−1.52484	−1.18599	−0.97035	−0.82106	−0.71157	−0.62787	−0.56179	−0.50832	−0.46444	−0.39665	−0.35071	−0.31823
−0.180	−0.99257	−0.59554	−0.42538	−0.33085	−0.27070	−0.22905	−0.98851	−0.17516	−0.15672	−0.14181	−0.12951	−0.11077	−0.09745	−0.08806
−0.160	−0.55649	−0.59554	−0.42538	−0.33085	−0.15177	−0.12842	−0.11130	−0.09821	−0.08787	−0.07950	−0.07261	−0.06261	−0.05459	−0.08915
−0.140	−0.39858	−0.23915	−0.17082	−0.13286	−0.10870	−0.09198	−0.07972	−0.07034	−0.06293	−0.05694	−0.05201	−0.04442	−0.03898	−0.03506
−0.10	−0.26180	−0.15709	−0.11221	−0.08727	−0.07140	−0.06082	−0.05236	−0.04620	−0.4134	−0.03740	−0.03416	−0.02911	−0.02548	−0.02305
−0.05	−0.18716	−0.11230	−0.08021	−0.06239	−0.05104	−0.04319	−0.03743	−0.03303	−0.02955	−0.02674	−0.02442	−0.02082	−0.01830	−0.01644
0	−0.14722	−0.08833	−0.06310	−0.04907	−0.04015	−0.03397	−0.02944	−0.02598	−0.02325	−0.02103	−0.01921	−0.01640	−0.01438	−0.01292
0.20	−0.08196	−0.04917	−0.03512	−0.02732	−0.02235	−0.01891	−0.01639	−0.01446	−0.01294	−0.01171	−0.10069	−0.00912	−0.00799	−0.00717
0.40	−.05795	−0.03477	−0.02484	−0.01932	−0.01580	−0.01337	−0.01159	−0.01023	−0.0915	−0.00828	−0.00756	−0.00645	−0.00565	−0.00506
0.60	−0.04528	−0.02717	−0.01940	−0.01509	−0.01235	−0.01045	−0.00906	−0.00799	−0.00715	−0.00647	−0.00591	−0.00504	−0.00441	−0.00395
0.80	−0.03738	−0.002243	−0.01602	−0.01246	−0.01020	−0.00863	−0.00748	−0.00660	−0.00590	−0.00534	−0.00488	−0.00416	−0.00364	−0.00326
1.0	−0.03197	−0.01918	−0.01370	−0.01066	−0.00872	−0.00738	−0.00639	−0.00568	−0.0505	−0.00457	−0.00417	−0.00356	−0.00311	−0.00279
1.2	−0.02802	−0.01681	−0.01201	−0.00934	−0.00764	−0.00647	−0.00560	−0.00494	−0.00442	−0.00400	−0.00365	−0.00312	−0.00273	−0.00243
1.4	−0.02499	−0.01499	−0.01071	−0.00833	−0.00682	−0.00577	−0.00500	−0.00441	−0.00395	−0.00357	−0.00326	−0.00278	−0.00243	−0.00218
1.8	−0.02065	−0.01239	−0.00885	−0.00689	−0.00563	−0 00477	−0.00413	−0.00365	−0.00326	−0.00295	−0.00269	−0.00230	−0.00201	−0.00180
2.0	−0.01904	−0.01143	−0.00816	−0.00635	−0.00519	−0.00439	−0.00381	−0.00336	−0.00301	−0.00272	−0.00248	−0.0212	−0.00185	−0.00166
2.4	−0.01652	−0.00991	−0.00708	−0.00551	−0.00451	−0.00381	−0.00330	−0.00292	−0.00261	−0.00236	−0.00216	−0.00184	−0.00160	−0.00144
3.0	−0.01385	−0.00831	−0.00594	−0.00462	−0.00378	−0.00320	−0.00277	−0.00244	−0.00219	−0.00198	−0.00181	−0.00154	−0.00135	−0.00120
4.0	−0.01098	−0.00660	−0.00471	−0.00367	−0.00300	−0.00254	−0.00220	−0.00194	−0.00174	−0.00157	−0.00144	−0.00122	−0.00107	−0.00096

Table 2 $T_{10}'(\beta,\epsilon',0)$

ϵ'/β	0	1	2	3	4	5	6	7	8	9	10	12	14	16
−0.195	12.67901	4.89667	2.00116	0.64231	−0.08113	−0.49651	−0.74609	−0.89972	−0.99329	−1.04859	−1.05534	−1.01978	−0.75505	−0.28120
−0.18	1.17613	0.45208	0.18440	0.05912	−0.00747	−0.04565	−0.06859	−0.08270	−0.09139	−0.09633	−0.09846	−0.09236	−0.07921	−0.06085
−0.16	0.40324	0.15404	0.06268	0.02007	−0.0025	−0.01508	−0.02375	−0.02802	−0.03097	−0.03268	−0.03336	−0.03245	−0.02801	−0.02297
−0.10	0.22073	0.08353	0.03391	0.01085	−0.00137	−0.00831	−0.01254	−0.01511	−0.01670	−0.01762	−0.01802	−0.01757	−0.01592	−0.01348
−0.05	0.05973	0.02222	0.00894	0.00285	−0.00036	−0.00218	−0.00327	−0.00393	−0.00780	−0.00826	−0.00847	−0.00857	−0.00808	−0.00668
0	0.04082	0.01503	0.00602	0.00191	−0.00026	−0.00146	−0.00219	−0.00263	−0.00438	−0.00459	−0.00471	−0.00477	−0.00445	−0.00406
0.2	0.01782	0.00638	0.00252	0.00080	−0.00010	−0.00060	−0.00090	−0.00108	−0.00291	−0.00307	−0.00316	−0.00319	−0.00308	−0.00287
0.4	0.01176	0.00414	0.00162	0.00051	−0.0006	−0.00039	−0.00057	−0.00069	−0.00119	−0.00126	−0.00130	−0.00135	−0.00137	−0.00137
0.6	0.00905	0.00315	0.00123	0.00039	−0.0005	−0.00029	−0.00043	−0.00052	−0.00076	−0.00080	−0.00083	−0.00087	−0.00091	−0.00096
0.8	0.00705	0.00259	0.00101	0.00032	−0.0005	−0.00024	−0.00035	−0.00042	−0.00057	−0.00060	−0.00062	−0.00066	−0.00070	−0.00074
1.0	0.00652	0.00226	0.00087	0.00027	−0.00003	−0.00020	−0.00030	−0.00036	−0.00046	−0.00049	−0.00051	−0.00054	−0.00058	−0.00062
1.2	0.00582	0.00199	0.00077	0.00024	−0.00003	−0.00018	−0.00027	−0.00032	−0.00040	−0.00042	−0.00043	−0.00046	−0.00049	−0.00056
1.4	0.00530	0.00180	0.00070	0.00026	−0.00003	−0.00016	−0.00024	−0.00029	−0.00035	−0.00037	−0.00038	−0.00041	−0.00044	−0.00046
1.8	0.00456	0.00154	0.00060	0.00019	−0.00002	−0.00014	−0.00021	−0.00025	−0.00032	−0.00033	−0.00035	−0.00037	−0.00040	−0.00044
2.0	0.00429	0.00145	0.00056	0.00017	−0.00002	−0.00013	−0.00017	−0.00023	−0.00027	−0.00028	−0.00029	−0.00031	−0.00035	−0.00038
2.4	0.00383	0.00130	0.00050	0.00016	−0.00002	−0.00012	−0.00017	−0.00020	−0.00025	−0.00037	−0.00027	−0.00029	−0.00032	−0.00035
3.0	0.00338	0.00114	0.00044	0.00014	−0.00002	−0.00010	−0.00015	−0.00018	−0.00023	−0.00024	−0.00025	−0.00026	−0.00027	−0.00032
4.0	0.00287	0.00097	0.00037	0.00012	−0.00001	−0.00009	−0.00013	−0.00015	−0.00020	−0.00021	−0.00022	−0.00022	−0.00025	−0.00028
									−0.0017	−0.00018	−0.00018	−0.00019	−0.00021	−0.00026

Table 3 $T_{11}'(\beta,\epsilon',0)$

ϵ'/β	0	1	2	3	4	5	6	7	8	9	10	12	14	16
−0.195	10.25980	5.51370	3.62497	2.63817	2.04212	1.64773	1.36983	1.16477	1.00803	0.88481	0.78575	0.63706	0.53149	0.45314
−0.18	4.40036	2.36485	1.85478	1.13153	0.87586	0.70671	0.58752	0.49957	0.43234	0.37950	0.33701	0.27324	0.22796	0.19435
−0.16	2.96704	1.59455	1.04834	0.76294	0.59057	0.47651	0.39614	0.33684	0.29151	0.25588	0.22723	0.18424	0.15370	0.13105
−0.14	2.36114	1.26893	0.83425	0.60714	0.46996	0.37920	0.31525	0.26805	0.23198	0.20363	0.18083	0.14661	0.12232	0.10429
−0.10	1.77297	0.95282	0.62643	0.45589	0.35289	0.28474	0.23671	0.20128	0.17419	0.15290	0.13578	0.11009	0.09185	0.07831
−0.05	1.41400	0.75990	0.49959	0.36359	0.28144	0.22709	0.18879	0.16053	0.13892	0.12194	0.10829	0.08780	0.07325	0.06245
0	1.20541	0.64779	0.42589	0.30995	0.23992	0.19359	0.16094	0.13685	0.11243	0.10396	0.09232	0.07485	0.06244	0.05324
0.2	0.82429	0.44299	0.29124	0.21196	0.16407	0.13238	0.11006	0.09358	0.08099	0.07109	0.06313	0.05118	0.04270	0.03641
0.4	0.66249	0.35604	0.23408	0.17035	0.13187	0.10640	0.08845	0.07521	0.06509	0.05114	0.05074	0.04114	0.03432	0.02926
0.6	0.56841	0.30548	0.20084	0.14616	0.11314	0.09129	0.07589	0.06453	0.05585	0.04902	0.04353	0.03529	0.02945	0.02511
0.8	0.50528	0.27155	0.17883	0.12993	0,10057	0.08115	0.06746	0.05736	0.04764	0.04358	0.03870	0.03138	0.02618	0.02232
1.0	0.45923	0.24680	0.16226	0.11809	0.09141	0.07375	0.061315	0.05214	0.04512	0.03961	0.03517	0.02852	0.02379	0.02028
1.2	0.42378	0.22775	0.14973	0.10897	0.08435	0.06806	0.05658	0.04811	0.04164	0.03655	0.03246	0.02631	0.02195	0.01872
1.4	0.39540	2.21250	0.13971	0.10167	0.07870	0.06350	0.08279	0.04489	0.03885	0.04310	0.03028	0.02455	0.02048	0.01746
1.8	0.35235	0.18936	0.12450	0.09060	0.07013	0.05659	0.04704	0.04000	0.03462	0.03039	0.02699	0.02188	0.01825	0.01556
2.0	0.33550	0.18030	0.11854	0.08627	0.06678	0.05388	0.04479	0.03809	0.03296	0.02893	0.02569	0.02083	0.01738	0.01482
2.4	0.30796	0.16549	0.10880	0.07918	0.06129	0.04946	0.04111	0.03496	0.03026	0.02565	0.02358	0.01912	0.01595	0.01360
3.0	0.27674	0.14883	0.09785	0.07121	0.055123	0.04448	0.03698	0.03144	0.02721	0.02388	0.02121	0.01720	0.01435	0.01223
4.0	0.241157	0.12960	0.08521	0.06201	0.04800	0.03873	0.03220	0.02730	0.02369	0.020797	0.01845	0.01497	0.01249	0.01065

The procedure for obtaining the shifted value of $\tilde{\beta}$ is given in Ref. 1. Once β is found, θ_1 can be found from Figs. 2-6. Note that Figs. 2-6 are plotted with $\epsilon' = 0$, 4, 8, 12, and 16. Values of ϵ' other than these can be obtained by interpolation since these ϵ' variation in $\theta'(\beta,\epsilon',0)$ is relatively smooth.

Numerical Example and Comparison for Flow Over Circular Cylinder

Let us consider the example of air ($Pr = 0.7$) flow over a cylinder with freestream velocity distribution given by Schmidt and Wenner[4] at $Re = 170{,}000$

$$U_e(x)/U_\infty = 3.631(x/2R) - 3.275(x/2R)^3 - 0.168(x/2R)^5$$

where R is the radius of the cylinder, and x is the circumferential distance measured from the forward stagnation point.

To demonstrate the computational procedure, we select for our example a location at $x/R = 0.5326$ on the cylinder. The values of ξ, $\tilde{\beta}$, ϵ, and ϵ' are first computed according to the transformations given in Ref. 1, (or see definitions in Nomenclature). The computations procedure is as follows: 1) $\tilde{\beta} = 0.375$, $\epsilon = -2.3597$, $\epsilon' = 5.4690$; Assume $\beta = 0.40$; 3) Δ_0 from Fig. 1 is 0.044; 4) Eq. (7) gives $\beta = 0.4792$; 5) Δ_0 from Fig. 1 is 0.0425; 6) Eq. (7) gives $\beta = 0.475$.

Further iteration yields essentially no change in β or Δ_0. With $\beta = 0.475$, we find that $f_0''(\beta, 0) = 0.91$, and $\theta_0'(\beta, 0)$ at $Pr = 0.7$ is -0.4678. By interpolation with Figs. 2-6, we find that $\theta_1'(\beta, \epsilon', 0) = -0.005$. Therefore, from Eq. (9) we get $\theta' = -0.4569$. Thus $(Nu_L/\sqrt{Re_L}) = 0.6973$.

A difference-differential numerical solution has been programmed by us for the purpose of comparison. The nondimensional shear stress is given in Fig. 7. The superiority of the present solution method over local similarity is obvious.

Results for the Nusselt coefficients are shown in Fig. 8. Here the local similarity solution does quite well. The series solution of Frössling which show considerable divergence for $x/2R > 0.5$. Figure 9 compares profile at $x/2R = 0.52336$.

Conclusion

The method of strained coordinates has been extended to treat the laminar thermal boundary layers with nonsimilarity arising from arbitrarily prescribed freestream velocity. Working charts and tables for a large range of Prandtl numbers are presented to facilitate computation, and a numerical example is given. The simplicity and accuracy of the present technique recommends its use in engineering calculations.

Appendix

We shall develop an asymptotic solution for higher Prandtl number in this Appendix. First we want to change the dependent variable η to $\zeta = (Pr)^{1/3}\eta$. Next we shall expand f_0, f_1, $(\partial f_0/\partial\beta)$, $(\partial\theta_0/\partial\beta)$, θ_0, and θ_1 in power series of $(Pr)^{-1/3}$ Particularly, we let

$$\theta_0(\zeta) = h_0(\zeta) + \left(\frac{1}{Pr}\right)^{1/3} h_1(\zeta) + \ldots \tag{A1}$$

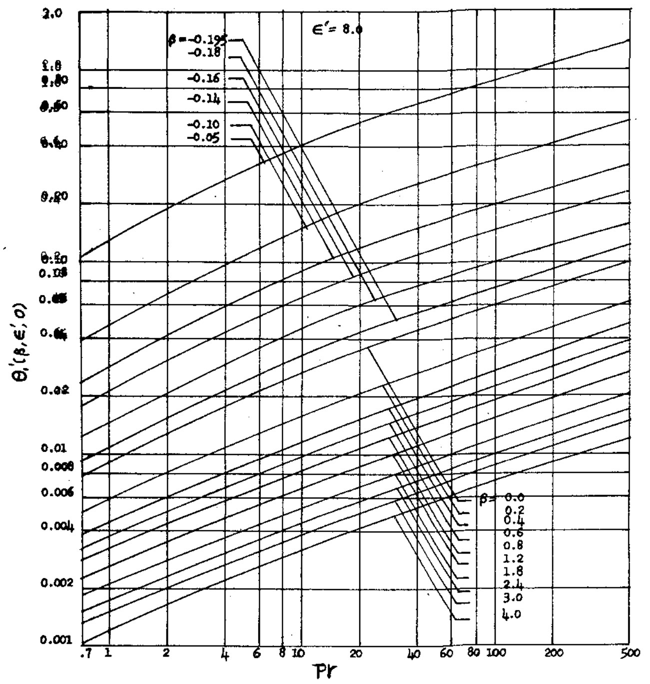

Fig. 4 $\theta_1'(\beta,\epsilon',0)$ vs Pr with $\epsilon' = 8$.

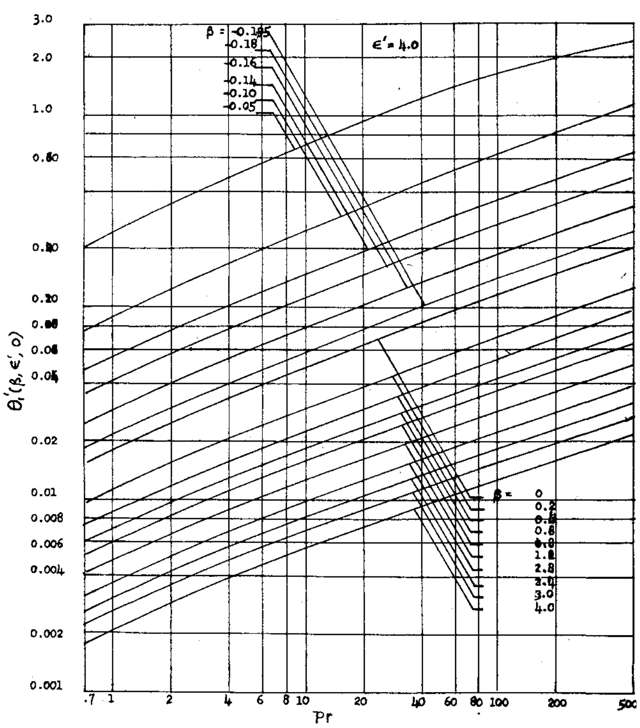

Fig. 3 $\theta_1'(\beta,\epsilon',0)$ vs Pr with $\epsilon' = 4$.

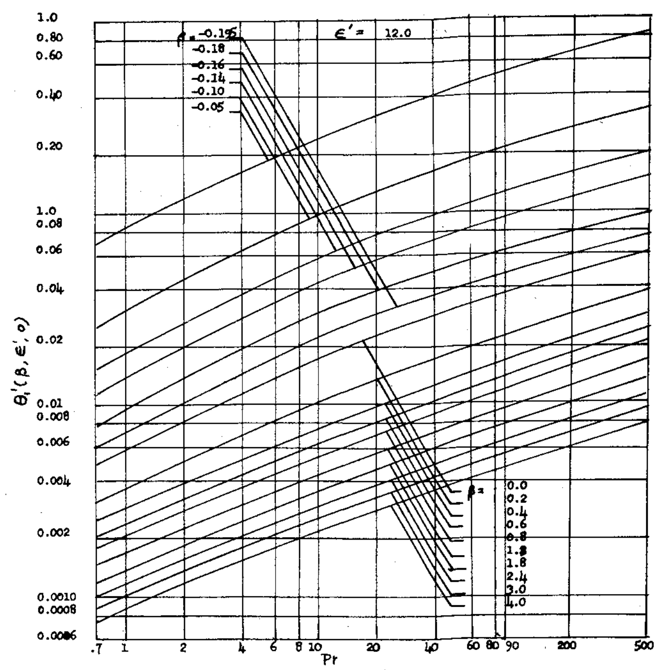

Fig. 5 $\theta_1'(\beta,\epsilon',0)$ vs Pr with $\epsilon' = 12$.

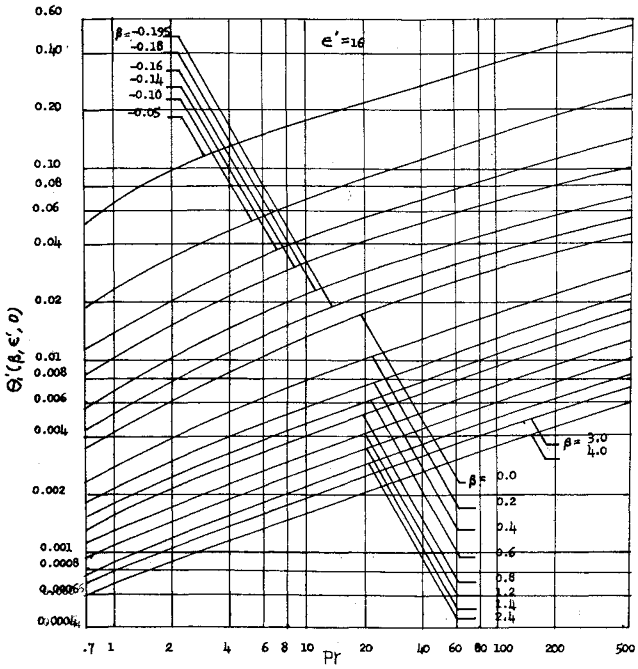

Fig. 6 $\theta_1'(\beta,\epsilon',0)$ vs Pr with $\epsilon'=16$.

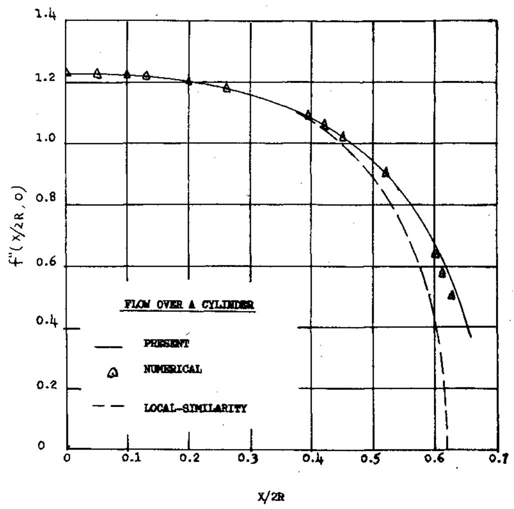

Fig. 7 Comparison of nondimensional shear stress for flow over a circular cylinder.

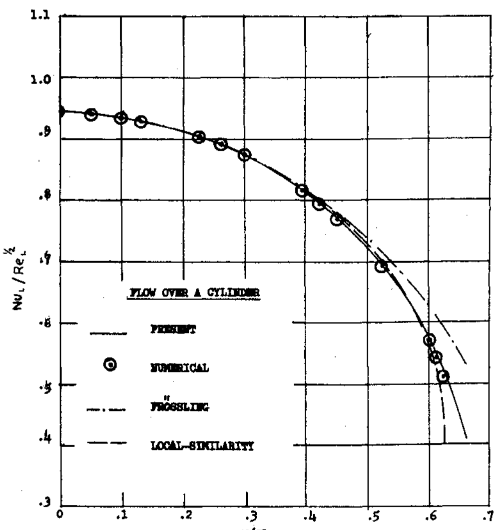

Fig. 8 Comparison of nondimensional heat transfer coefficients.

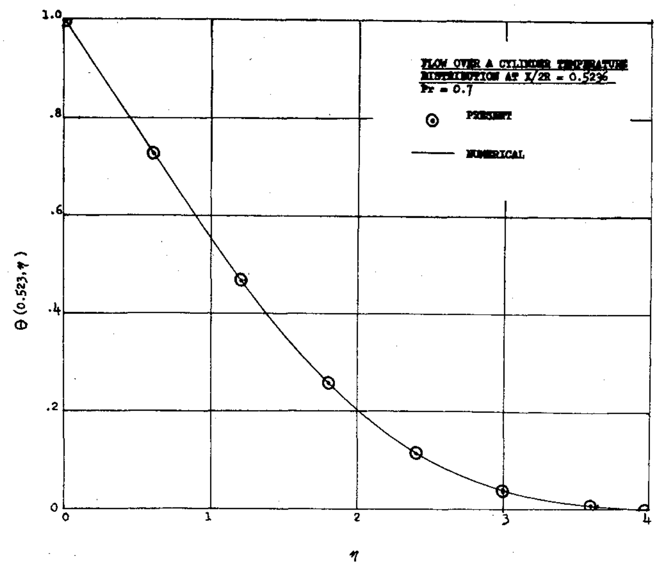

Fig. 9 Comparison of nondimensional temperature profile at $x/2R$ = 0.5236.

$$\theta_1(\zeta)=(1-\epsilon'\Delta_0)\left[T_0(\zeta)+\left(\frac{1}{Pr}\right)^{1/3}T_1(\zeta)+\ldots\right] \tag{A2}$$

The procedure for obtaining $\theta_0(\zeta)$ here as $Pr\rightarrow\infty$ is essentially the same as that given by Morgan and Warner.[6]

Upon substituting Eq. (A1) into Eq. (10, 11) and equating terms of powers of $(1/Pr)^{1/3}$, we find that

$$h_0''+\frac{a_2}{2}\zeta^2h_0'=0 \tag{A3}$$

$$h_0(0)=1, \qquad h_0(\infty)=0 \tag{A4}$$

$$h_1''+\frac{a_2}{2}\zeta^2h_1'=\frac{\beta}{3!}\zeta^2h_0 \tag{A5}$$

$$h_1(0)=h_1(\infty)=0 \tag{A6}$$

where

$$a_2=f_0''(\beta,0)$$

The solutions for h_0 and h_1 can be put into closed forms, and are given by

$$h_0(\zeta)=1-\frac{\Gamma\left(1/3,\frac{a_2}{b}\zeta^3\right)}{\Gamma\left(\frac{1}{3}\right)} \tag{A7}$$

$$h_1(\zeta)=\frac{-\beta}{4\Gamma\left(\frac{1}{3}\right)a_2}\left(\frac{6}{a_2}\right)^{1/3}\left\{\Gamma\left(\frac{5}{3},\frac{a_2}{6}\zeta^3\right)-\frac{\Gamma\left(\frac{5}{3}\right)}{\Gamma\left(\frac{1}{3}\right)}\Gamma\left(\frac{1}{3},\frac{a_2}{6}\zeta^3\right)\right\} \tag{A8}$$

Thus

$$\theta_0'(\beta,0)=-Pr^{1/3}\left\{0.776458\left(\frac{a_2}{2}\right)^{1/3}-0.094310\left(\frac{1}{Pr}\right)^{1/3}\frac{\beta}{a_2}\right\} \tag{A9}$$

Table 4 Comparison for $\left[\dfrac{\theta_1'(0)\ \text{Asymptotic}}{\theta_1'(0)\ \text{Numerical}}\right] Pr=10$

ϵ'/β	0	2	4	8
-0.160	$\frac{-0.6510}{-0.8259}$	$\frac{-0.2403}{-0.2658}$	$\frac{-0.1445}{-0.1485}$	$\frac{-0.0778}{-0.0918}$
0	$\frac{-0.2414}{-0.2476}$	$\frac{-0.0814}{-0.0842}$	$\frac{-0.0456}{-0.0473}$	$\frac{-0.0226}{-0.0259}$
2.0	$\frac{-0.0353}{-0.0341}$	$\frac{-0.0147}{-0.0146}$	$\frac{-0.0090}{-0.0090}$	$\frac{-0.0049}{-0.0049}$

A comparison between Eq. (A9) and numerical solutions given in Ref. 2 shows that this asymptotic expression is highly accurate. $T_0(\zeta)$ is found to satisfy the following equation

$$T_0'' + \frac{a_2}{2}\zeta^2 T_0' - \epsilon' a_2 \zeta T_0 = \left(a_2 \zeta \frac{\partial h_0}{\partial \beta} - \frac{b_2}{2}\zeta^2 h_0'\right) \tag{A10}$$

$$T_0(0) = T_0(\infty) = 0 \tag{A11}$$

where

$$b_2 = \frac{\partial f_0''}{\partial \beta}(\beta, 0)$$

T_1 can be split into two parts, namely

$$T_1(\zeta) = T_{10}(\zeta) + \frac{(1+\epsilon')}{(1-\epsilon'\Delta_0)}\frac{\Delta_0}{6} T_{11}(\zeta) \tag{A12}$$

so that $T_{10}(\zeta)$ and $T_{11}(\zeta)$ satisfy the following

$$T_{10}'' + \frac{a_2}{2}\zeta^2 T_{10}' - \epsilon' a_2 \zeta T_{10} = \frac{-\beta}{2}\zeta^2 \left\{ \frac{\partial h_0}{\partial \beta} + \epsilon' T_0 - \frac{\zeta}{3} T_0' \right\} + a_2 \zeta \frac{\partial h_1}{\partial \beta} + \frac{1}{3!}\zeta^3 h_0' - \frac{b_2}{2} h_1' \tag{A13}$$

$$T_{11}'' + \frac{a_2}{2}\zeta^2 T_{11}' - \epsilon' a_2 \zeta T_{11} = \zeta^3 h_0' \tag{A14}$$

$$T_{10}(0) = T_{10}(\infty) = T_{11}(0) = T_{11}(\infty) = 0 \tag{A15}$$

The solutions for Eqs. (A10-A15) are obtained numerically. Particularly, $T_0'(0)$, $T_{10}'(10)$, and $T_{11}'(0)$ for various combination of ϵ' and β are given in Tables 1-3.

The asymptotic solution for $\theta_1'(\beta, \epsilon', 0)$ as $Pr \to \infty$ is given by

$$\theta_1'(\beta,\epsilon',0) = Pr^{1/3}(1-\epsilon'\Delta_0)\left\{T_0'(\beta,\epsilon',0) + \left(\frac{1}{Pr}\right)^{1/3} \times \left[T_{10}'(\beta,\epsilon',0) + \frac{(1+\epsilon')}{(1-\epsilon'\Delta_0)}\frac{\Delta_0}{6} T_{11}'(\beta,\epsilon',0)\right]\right\} \tag{A16}$$

Comparison between the numerical solution and Eq. A16 given in Table 4 for the case of $Pr = 10$, shows that the asymptotic formula is quite accurate.

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